HOMOGENEOUS REAL HYPERSURFACES OF TYPE (A_2) IN A COMPLEX PROJECTIVE SPACE

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Contents

1. Preface

In Section 2, we prepare basic terminologies on real hypersurfaces and some fundamental properties of Hopf hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$ of constant holomorphic sectional curvature $c(\neq 0)$, namely $\widetilde{M}_n(c)$ is congruent to either a complex n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c(>0)$ or a complex n-dimensional complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature $c(<0)$.

In the theory of real hypersurfaces isometrically immersed into $\widetilde{M}_n(c)$, homogeneous Hopf hypersurfaces are fundamental. In Sections 3 and 4, we review the classification theorems of homogeneous Hopf hypersurfaces M in $\widetilde{M}_n(c)$. In $\mathbb{C}P^n(c)$, we note that every homogeneous real hypesurface is automatically a Hopf hypersurface in this space (see [14]). However, in $\mathbb{C}H^n(c)$ there exist many homogeneous real hypersurfaces which are *not* Hopf hypersurfaces in this ambient space (cf. [5]).

Among real hypersurfaces in $\mathbb{C}P^n(c)$ the following hypersurfaces are typical examples:

(A_1) A geodesic sphere of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$;
(A_2) A tube of radius r $(0 < r < \pi/\sqrt{c}\,)$ around a totally geodesic Kähler submanifold $\mathbb{C}P^\ell(c)\,(1 \leqq \ell \leqq n-2)$ in $\mathbb{C}P^n(c)$.

These real hypersurfaces are said to be of type (A_1) and of type (A_2), respectively. In this paper, we pay particular attention to real hypersurfaces of types (A_1) and (A_2) in a complex projective space. The following theorem shows the importance of these hypersurfaces.

Theorem A ([12]). *For each real hypersurface M^{2n-1} of $\mathbb{C}P^n(c), n \geqq 2$, the length of the derivative of the shape operator A of M satisfies $\|\nabla A\|^2 \geqq c^2(n-1)/4$. The*

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equality holds on M if and only if M is locally congruent to one of real hypersurfaces of type (A_1) *and type* (A_2).

Real hypersurfaces of type (A_1) have two distinct constant principal curvatures in $\mathbb{C}P^n(c)$. It is well-known that $\mathbb{C}P^n(c)$ does *not* admit totally umbilic real hypersurfaces and that a real hypersurface M^{2n-1} of $\mathbb{C}P^n(c), n \geqq 3$ is of type (A_1) if and only if M has at most two distinct principal curvatures at each point of M. These imply that real hypersurfaces of type (A_1) are the simplest examples of real hypersurfaces in $\mathbb{C}P^n(c)$ and that there exist no real hypersurfaces M all of whose geodesics are mapped to circles in $\mathbb{C}P^n(c)$.

We here recall the following characterization of real hypersurfaces of type (A_1) in $\mathbb{C}P^n(c)$ from this viewpoint.

Theorem B ([10])**.** *A connected real hypersurface M^{2n-1} of $\mathbb{C}P^n(c), n \geqq 2$ is locally congruent to a real hypersurface of type* (A_1) *of radius r* $(0 < r < \pi/\sqrt{c}\,)$ *if and only if there exist orthonormal vectors $v_1, v_2, \ldots, v_{2n-2}$ perpendicular to the characteristic vector ξ_x at each point $x \in M$ satisfying the following two conditions*:

- (*i*) *All geodesics $\gamma_i = \gamma_i(s)$ on M^{2n-1} with $\gamma_i(0) = x$ and $\dot\gamma_i(0) = v_i$* $(1 \leqq i \leqq 2n-2)$ *are mapped to circles of positive curvature in $\mathbb{C}P^n(c)$*;
- (*ii*) *All geodesics $\gamma_{ij} = \gamma_{ij}(s)$ on M^{2n-1} with $\gamma_{ij}(0) = x$ and $\dot\gamma_{ij}(0) = (v_i+v_j)/\sqrt{2}$* $(1 \leqq i < j \leqq 2n-2)$ *are mapped to circles of positive curvature in $\mathbb{C}P^n(c)$.*

In Section 5, we explain the importance of hypersurfaces of type (A) in detail.

In Section 6, we characterize *minimal* real hypersurfaces M^{2n-1} of type (A_2) in a complex projective space by observing some geodesics on M in the class of all *minimal* real hypersurfaces from the viewpoint of Theorem B (see Theorem). Note that there do *not* exist minimal real hypersurfaces M^{2n-1} of type (A_2) in a complex hyperbolic space.

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2. Basic terminologies and preliminaries

Let M^{2n-1} be a real hypersurface immersed into a nonflat complex space form $\widetilde{M}_n(c)$ through an isometric immersion with a unit normal local vector field $\mathcal{N}$. The Riemannian connections $\widetilde\nabla$ of $\widetilde{M}_n(c)$ and ∇ of M are related by the following formulas of Gauss and Weingarten:

$$\widetilde\nabla_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}, \tag{2.1}$$

$$\widetilde\nabla_X \mathcal{N} = -AX \tag{2.2}$$

for arbitrary vector fields X and Y on M, where g is the Riemannian metric of M induced from the standard metric of the ambient space $\widetilde{M}_n(c)$ and A is the shape operator of M in $\widetilde{M}_n(c)$. An eigenvector of the shape operator A is called a *principal curvature vector* of M in $\widetilde{M}_n(c)$ and an eigenvalue of A is called a *principal curvature* of M in $\widetilde{M}_n(c)$. We set $V_\lambda = \{v \in TM \mid Av = \lambda v\}$ which is called the principal distribution associated to the principal curvature λ.

It is well-known that M has an almost contact metric structure induced from the Kähler structure (J, g) of the ambient space $\widetilde{M}_n(c)$. That is, we have a quadruple (ϕ, ξ, η, g) defined by

$$g(\phi X, Y) = g(JX, Y), \quad \xi = -J\mathcal{N} \quad \text{and} \quad \eta(X) = g(\xi, X) = g(JX, \mathcal{N}).$$

Then they satisfy

$$\phi^2(X) = -X + \eta(X)\xi, \ \eta(\xi) = 1 \quad \text{and} \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vectors $X, Y \in TM$. It is known that these equations imply that $\phi\xi = 0$ and $\eta(\phi(X)) = 0$. In the following, we call ϕ, ξ and η *the structure tensor*, *the characteristic vector* and *the contact form* on M, respectively.

It follows from (2.1), (2.2), $\widetilde{\nabla} J = 0$ and $JX = \phi X + \eta(X)\mathcal{N}$ that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.3}$$

$$\nabla_X \xi = \phi AX. \tag{2.4}$$

Indeed, for the first equality, we get

$$\begin{aligned}\nabla_X \xi &= -\nabla_X(J\mathcal{N}) = -\widetilde{\nabla}_X(J\mathcal{N}) + g(AX, J\mathcal{N})\mathcal{N} \\ &= -J\widetilde{\nabla}_X \mathcal{N} + g(AX, J\mathcal{N})\mathcal{N} = JAX - g(JAX, \mathcal{N})\mathcal{N} = \phi AX.\end{aligned}$$

For the second, we see

$$\begin{aligned}(\nabla_X \phi)Y &= \nabla_X(\phi Y) - \phi\nabla_X Y = \nabla_X(JY - \eta(Y)\mathcal{N}) - \phi\nabla_X Y \\ &= \widetilde{\nabla}_X(JY - \eta(Y)\mathcal{N}) - g(A\phi Y, X)\mathcal{N} - \phi\nabla_X Y \\ &= J(\nabla_X Y + g(AX, Y)\mathcal{N}) - X(\eta(Y))\xi + \eta(Y)AX \\ &\qquad - g(A\phi Y, X)\mathcal{N} - \phi\nabla_X Y \\ &= \phi\nabla_X Y + g(\nabla_X Y, \xi)\mathcal{N} - g(AX, Y)\xi - g(\nabla_X Y, \xi)\mathcal{N} \\ &\quad - g(Y, \phi AX)\mathcal{N} + \eta(Y)AX - g(A\phi Y, X)\mathcal{N} - \phi\nabla_X Y \\ &= \eta(Y) - g(AX, Y)\xi.\end{aligned}$$

Denoting the curvature tensor of M by R, we have the equation of Gauss given by

$$\begin{aligned}g((R(X, Y)Z, W) = (c/4)\{&g(Y, Z)g(X, W) \\ &- g(X, Z)g(Y, W) + g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) \\ &- 2g(\phi X, Y)g(\phi Z, W)\} \\ &+ g(AY, Z)g(AX, W) - g(AX, Z)g(AY, W).\end{aligned} \tag{2.5}$$

The following is called the equation of Codazzi.

$$(\nabla_X A)Y - (\nabla_Y A)X = (c/4)(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi), \tag{2.6}$$

Let K be the sectional curvature of M. That is, K is defined by $K(X, Y) = g(R(X, Y)Y, X)$, where X and Y are orthonormal vectors on M. Then it follows from (2.5) that

$$K(X, Y) = (c/4)\big(1 + 3g(\phi X, Y)^2\big) + g(AX, X)g(AY, Y) - g(AX, Y)^2. \tag{2.7}$$

We usually call M a *Hopf hypersurface* if the characteristic vector ξ of M is a principal curvature vector at each point of M. The following lemma clarifies fundamental properties of principal curvatures of a Hopf hypersurface M in $\widetilde{M}_n(c)$.

Lemma 1 ([12, 9]). *Let M be a Hopf hypsurface of a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$. Then the following hold.*

(1) *If a nonzero vector $v \in TM$ orthogonal to ξ satisfies $Av = \lambda v$, then $(2\lambda - \delta)A\phi v = (\delta\lambda + (c/2))\phi v$, where δ is the principal curvature associated with ξ. In particular, when $c > 0$, we have $A\phi v = \big((\delta\lambda + (c/2))/(2\lambda - \delta)\big)\phi v$.*

(2) *The principal curvature δ associated with ξ is locally constant.*

Proof. We adopt the discussion in the proof of this lemma in [13].

(1) It follows from (2.4) and $A\xi = \delta\xi$ that

$$(\nabla_X A)\xi = \nabla_X(A\xi) - A\nabla_X\xi = (X\delta)\xi + (\delta I - A)\phi AX. \tag{2.8}$$

This, together with (2.6), shows

$$X\delta = g((\nabla_X A)\xi, \xi) = g((\nabla_\xi A)X, \xi). \tag{2.9}$$

So, from $g((\nabla_\xi A)X, \xi) = g((\nabla_\xi A)\xi, X) = (\xi\delta)\eta(X)$ we see that $X\delta = 0$ for all vectors X perpendicular to ξ, so that grad $\delta = (\xi\delta)\xi$. Now, using (2.8) and (2.9), we have

$$\begin{aligned} g((\nabla_X A)Y, \xi) &= g((\nabla_X A)\xi, Y) \\ &= (\xi\delta)\eta(X)\eta(Y) + g((\delta I - A)\phi AX, Y). \end{aligned} \tag{2.10}$$

Exchanging X and Y in (2.10) and subtracting these equations, we compute

$$g((\nabla_X A)Y, \xi) - g((\nabla_Y A)X, \xi) = g((\delta I - A)\phi AX, Y) - g((\delta I - A)\phi AY, X).$$

This, combined with (2.6), implies

$$\begin{aligned} (c/2)g(X, \phi Y) &= g((\delta I - A)\phi AX, Y) - g((\delta I - A)\phi AY, X) \\ &= -g(X, A\phi(\delta I - A)Y) - g(X, (\delta I - A)\phi AY) \end{aligned}$$

for all $X, Y \in TM$. Thus we can see that

$$A\phi A - (\delta/2)(A\phi + \phi A) - (c/4)\phi = 0. \tag{2.11}$$

Statement (1) is an immediate consequence of (2.11).

(2) Let $\beta = \xi\delta$. Then grad $\delta = \beta\xi$ (see the proof of Statement (1)). We have

$$\begin{aligned} &g(\nabla_X(\text{grad } \delta), Y) - g(\nabla_Y(\text{grad } \delta), X) \\ &\quad = X(g(\text{grad } \delta, Y)) - g(\text{grad } \delta, \nabla_X Y) - Y(g(\text{grad } \delta, X)) \\ &\qquad\quad + g(\text{grad } \delta, \nabla_Y X) \\ &\quad = XY\delta - YX\delta - g(\text{grad } \delta, \nabla_X Y - \nabla_Y X) \\ &\quad = ([X, Y] - (\nabla_X Y - \nabla_Y X))\delta = 0. \end{aligned}$$

This, together $\beta = \xi\delta$, yields

$$\begin{aligned} 0 &= g(\nabla_X(\beta\xi), Y) - g(\nabla_Y(\beta\xi), X) \\ &= X\beta\eta(Y) + \beta g(\phi AX, Y) - Y\beta\eta(X) - \beta g(\phi AY, X) \\ &= (X\beta)\eta(Y) - (Y\beta)\eta(X) + \beta g((\phi A + A\phi)X, Y). \end{aligned} \tag{2.12}$$

Setting $Y = \xi$ in (2.12), we get $0 = X\beta - (\xi\beta)\eta(X)$, where we have used $A\xi = \delta\xi$ and $\phi\xi = 0$. Thus we see that $X\beta = (\xi\beta)\eta(X)$ for all vectors X. This, combined with (2.12), shows

$$(\xi\delta)(\phi A + A\phi) = 0. \tag{2.13}$$

Note that Equation (2.13) is a key in the proof of Statement (2). In the following, we suppose that $\xi\delta \neq 0$ at some point. Then it follows from (2.13) that $\phi A + A\phi = 0$ in a sufficiently small neighborhood of this point. So, from (2.11) we know that $\phi A^2 + (c/4)\phi = 0$. Now, applying this equation to a principal curvature vector X orthogonal to ξ, we get

$$0 = \phi(A^2 + (c/4)I)X = (\lambda^2 + (c/4))\phi X,$$

where λ is the principal curvature for X. Hence $\lambda^2 + (c/4) = 0$. Then we obtain a contradiction in the case of $c > 0$. Thus we find that grad $\delta = 0$, namely δ is constant locally on M when c is positive.

Therefore the rest of the proof is to verify that $\xi\delta = 0$ also holds on M in the case of $c < 0$. Suppose that $\phi A + A\phi = 0$. So we can use $\phi(A^2 + (c/4)I) = 0$. Hence

$$\begin{aligned} 0 &= (\nabla_X(\phi(A^2 + (c/4)I)))Y \\ &= (\nabla_X\phi)(A^2 + (c/4)I)Y + \phi(\nabla_X A)AY + \phi A(\nabla_X A)Y. \end{aligned} \tag{2.14}$$

Hence, from (2.3), Equation (2.14) becomes

$$\begin{aligned} 0 = (\delta^2 + (c/4))\eta(Y)AX - g((A^3 + (c/4)A)X, Y)\xi + \phi(\nabla_X A)AY \\ + \phi A(\nabla_X A)Y. \end{aligned}$$

Applying ϕ to this equality, we get

$$\phi((\delta^2 + (c/4))\eta(Y)AX) + \phi^2((\nabla_X A)AY) + \phi^2(A(\nabla_X A)Y) = 0. \tag{2.15}$$

The second term of (2.15) is rewritten as

$$\phi^2((\nabla_X A)AY) = -(\nabla_X A)AY + g((\nabla_X A)AY, \xi)\xi.$$

It follows from (2.4), $g((\nabla_X A)Y, Z) = g(Y, (\nabla_X A)Z)$ and $\phi A^2 = -(c/4)\phi$ that

$$\begin{aligned} g((\nabla_X A)AY, \xi) &= g(AY, (X\delta)\xi + (\delta I - A)\phi AX) \\ &= \delta\beta\eta(X)\eta(Y) + \delta g(A\phi AX, Y) - g(A^2\phi AX, Y) \\ &= \delta\beta\eta(X)\eta(Y) + (c/4)\delta g(\phi X, Y) + (c/4)g(\phi AX, Y). \end{aligned}$$

Again using $\phi^2 X = -X + g(X, \xi)\xi$, we can rewrite the third term of (2.15) as

$$\phi^2(A(\nabla_X A)Y) = -A(\nabla_X A)Y + g(A(\nabla_X A)Y, \xi)\xi,$$

and by a direct computation we see that

$$\begin{aligned} g(A(\nabla_X A)Y, \xi) &= \delta g((\nabla_X A)Y, \xi) \\ &= \delta\beta\eta(X)\eta(Y) + \delta^2 g(\phi AX, Y) - (c\delta/4)g(\phi X, Y). \end{aligned}$$

Then by all of the above computation we know that

$$\begin{aligned} (\nabla_X A)AY + A(\nabla_X A)Y = 2\delta\beta\eta(X)\eta(Y)\xi \\ + (\delta^2 + (c/4))(g(\phi AX, Y)\xi + \eta(Y)\phi AX). \end{aligned} \tag{2.16}$$

Here, exchanging X and Y in (2.16) and subtracting these equations, from (2.6) and the equality $\phi A + A\phi = 0$ we know that

$$(\nabla_X A)AY - (\nabla_Y A)AX = (c\delta/2)g(\phi X, Y)\xi + \delta^2(\eta(Y)\phi AX - \eta(X)\phi AY). \tag{2.17}$$

Taking the inner product of $(\nabla_X A)AY$ and Z, from the symmetry of A, $\phi A + A\phi = 0$ and (2.6) we have

$$\begin{aligned} g((\nabla_X A)AY, Z) &= g(AY, (\nabla_X A)Z) \\ &= g(AY, (\nabla_Z A)X) \\ &\quad + (c/4)(\eta(X)g(A\phi Y, Z) - \eta(Z)g(A\phi X, Y) + 2\delta\eta(Y)g(X, \phi Z)). \end{aligned}$$

Exchanging X and Y in this equation and subtracting the two equations, we obtain

$$\begin{aligned} g((\nabla_X A)AY, Z) - g((\nabla_Y A)AX, Z) &= g(AY, (\nabla_Z A)X) \\ &- g(AX, (\nabla_Z A)Y) + (c/4)(\eta(X)g(A\phi Y, Z) - \eta(Y)g(A\phi X, Z)) \\ &+ (c\delta/2)(\eta(Y)g(X, \phi Z) - \eta(X)g(Y, \phi Z)). \end{aligned}$$

Then the coefficient of X on the right hand side of this equation is

$$\begin{aligned} &(\nabla_Z A)AY - A(\nabla_Z A)Y + (c/4)(g(A\phi Y, Z)\xi - \eta(Y)A\phi Z) \\ &\qquad + (c\delta/2)(\eta(Y)\phi Z - g(Y, \phi Z)\xi). \end{aligned} \tag{2.18}$$

On the other hand, taking the inner product of (2.17) and Z, we find that the coefficient of X on the right hand side is

$$-(c\delta/2)\eta(Z)\phi Y + \delta^2(\eta(Y)\phi AZ - g(\phi AY, Z)\xi).$$

This, together with (2.18), yields

$$\begin{aligned} (\nabla_Z A)AY - A(\nabla_Z A)Y &= (\delta^2 - (c/4))\eta(Y)\phi AZ \\ &- (c\delta/2)(\eta(Y)\phi Z + \eta(Z)\phi Y + g(\phi Y, Z)\xi) - (\delta^2 - (c/4))g(\phi AY, Z)\xi. \end{aligned}$$

Replacing Z with X in this equation, we have

$$\begin{aligned} (\nabla_X A)AY - A(\nabla_X A)Y &\\ = (\delta^2 - (c/4))(\eta(Y)\phi AX - g(\phi AX, Y)\xi) &\\ - (c\delta/2)(\eta(Y)\phi X + \eta(X)\phi Y + g(\phi Y, X)\xi). & \end{aligned} \tag{2.19}$$

It follows from (2.16) and (2.19) that

$$\begin{aligned} (\nabla_X A)AY = \beta\delta\eta(X)\eta(Y)\xi + (c/4)g(\phi AX, Y)\xi + \delta^2\eta(Y)\phi AX &\\ - (c\delta/4)(\eta(Y)\phi X + \eta(X)\phi Y + g(\phi Y, X)\xi). & \end{aligned} \tag{2.20}$$

Also recall that $A\phi A = (c/4)\phi$. Replacing Y by AY in (2.20), we get

$$\begin{aligned} (\nabla_X A)A^2Y &= \beta\delta^2\eta(X)\eta(Y)\xi + (c^2/16)g(\phi X, Y)\xi \\ &+ \delta^3\eta(Y)\phi AX - (c\delta^2/4)\eta(Y)\phi X - (c\delta/4)\eta(X)\phi AY \\ &- (c\delta/4)g(\phi AY, X)\xi. \end{aligned} \tag{2.21}$$

We note that $(A^2 + (c/4)I)Y = (\delta^2 + (c/4))\eta(Y)\xi$, since $\phi(A^2 + (c/4)I) = 0$. This shows that $A^2Y = (-c/4)Y + (\delta^2 + (c/4))\eta(Y)\xi$. So we can compute directly the following equalities.

$$\begin{aligned}(\nabla_X A)A^2Y &= (-c/4)(\nabla_X A)Y + (\delta^2 + (c/4))\eta(Y)(\nabla_X A)\xi \\ &= (-c/4)(\nabla_X A)Y + (\delta^2 + (c/4))\beta\eta(Y)\eta(X)\xi \\ &\quad + (\delta^2 + (c/4))\eta(Y)\delta\phi AX - (\delta^2 + (c/4))(c/4)\eta(Y)\phi X \\ &= (-c/4)(\nabla_X A)Y + \beta\delta^2\eta(X)\eta(Y)\xi + (c\beta/4)\eta(X)\eta(Y)\xi \\ &\quad + \delta^3\eta(Y)\phi AX + (c\delta/4)\eta(Y)\phi AX - (c\delta^2/4)\eta(Y)\phi X - (c^2/16)\eta(Y)\phi X.\end{aligned}$$

This, combined with (2.21), shows

$$\begin{aligned}(\nabla_X A)Y = \beta\eta(X)\eta(Y)\xi + \delta(\eta(X)\phi AY + \eta(Y)\phi AX \\ + g(\phi AX, Y)\xi) + (c/4)(g(\phi Y, X)\xi - \eta(Y)\phi X).\end{aligned} \tag{2.22}$$

We shall compute $(R(X, \phi X) \cdot A)Z$ for each X orthogonal to ξ by using (2.22), which is defined by

$$(R(X, \phi X) \cdot A)Z = R(X, \phi X)(AZ) - A(R(X, \phi X)Z),$$

where R is the curvature tensor of our real hypersurface M. By a direct calculation we find

$$\begin{aligned}&\nabla_X((\nabla_{\phi X} A)Z) \\ &= \nabla_X(\delta(\eta(Z)\phi A\phi X + g(\phi A\phi X, Z)\xi) + (c/4)(g(\phi Z, \phi X)\xi - \eta(Z)\phi^2 X)) \\ &= \delta(g(\nabla_X Z, \xi)AX + g(Z, \nabla_X \xi)AX + \eta(Z)\nabla_X(AX) \\ &\qquad + g(\nabla_X(AX), Z)\xi + g(AX, \nabla_X Z)\xi + g(AX, Z)\nabla_X \xi) \\ &\quad + (c/4)(g(\nabla_X X, Z)\xi + g(X, \nabla_X Z)\xi + g(X, Z)\nabla_X \xi \\ &\qquad + g(\nabla_X \xi, Z)X + g(\xi, \nabla_X Z)X + \eta(Z)\nabla_X X),\end{aligned} \tag{2.23}$$

where we have used $X\delta = \eta(X)\beta$. Here, from (2.22) we see that

$$\nabla_X(AX) = A(\nabla_X X) + (\nabla_X A)X = A(\nabla_X X) + \delta g(\phi AX, X)\xi.$$

Then we rewrite (2.23) as

$$\begin{aligned}&\nabla_X((\nabla_{\phi X} A)Z) \\ &= \delta(g(\nabla_X Z, \xi)AX + g(Z, \phi AX)AX + \eta(Z)A(\nabla_X X) \\ &\quad + \delta g(\phi AX, X)\eta(Z)\xi + g(A(\nabla_X X), Z)\xi + \delta g(\phi AX, X)\eta(Z)\xi \\ &\qquad + g(AX, \nabla_X Z)\xi + g(AX, Z)\phi AX) \\ &\quad + (c/4)(g(\nabla_X X, Z)\xi + g(X, \nabla_X Z)\xi + g(X, Z)\phi AX \\ &\qquad + g(\phi AX, Z)X + g(\xi, \nabla_X Z)X + \eta(Z)\nabla_X X).\end{aligned} \tag{2.24}$$

Moreover, we have similarly

$$\begin{aligned}(\nabla_{\phi X} A)(\nabla_X Z) = \delta(g(\nabla_X Z, \xi)AX + g(AX, \nabla_X Z)\xi) \\ + (c/4)(g(\nabla_X Z, X)\xi + g(\xi, \nabla_X Z)X)\end{aligned} \tag{2.25}$$

and

$$(\nabla_{\nabla_X \phi X} A)Z \\ = -\beta g(AX, X)\eta(Z)\xi + \delta(\eta(Z)A\nabla_X X - g(AX, X)\phi AZ \\ - \eta(Z)g(\nabla_X X, \xi)\delta\xi + g(A\nabla_X X, Z)\xi - \delta g(\nabla_X X, \xi)\eta(Z)\xi) \\ + (c/4)(g(Z, \nabla_X X)\xi - \eta(Z)g(\nabla_X X, \xi)\xi + \eta(Z)\nabla_X X \\ - \eta(Z)g(\nabla_X X, \xi)\xi). \tag{2.26}$$

We now define

$$N(X, Z) = (\nabla_X \nabla_{\phi X} A - \nabla_{\nabla_X \phi X} A)Z \\ = \nabla_X((\nabla_{\phi X} A)Z) - (\nabla_{\phi X} A)(\nabla_X Z) - (\nabla_{\nabla_X \phi X} A)Z.$$

This, together with (2.24), (2.25) and (2.26), implies

$$N(X, Z) = \beta g(AX, X)\eta(Z)\xi \\ + \delta(g(Z, \phi AX)AX + g(AX, Z)\phi AX + g(AX, X)\phi AZ) \\ + (c/4)(g(X, Z)\phi AX + g(\phi AX, Z)X - 2\eta(Z)g(\phi AX, X)\xi). \tag{2.27}$$

Since X is perpendicular to ξ, by the definition of N we get

$$N(\phi X, Z) = (-\nabla_{\phi X} \nabla_X A + \nabla_{\nabla_{\phi X} X} A)Z \\ = (R(X, \phi X) \cdot A)Z - (\nabla_X \nabla_{\phi X} A - \nabla_{\nabla_X \phi X} A)Z,$$

so that

$$(R(X, \phi X) \cdot A)Z = N(X, Z) + N(\phi X, Z).$$

On the other hand, from (2.27) we know that

$$N(\phi X, Z) = -\beta g(AX, X)\eta(Z)\xi \\ + \delta(g(Z, AX)A\phi X + g(A\phi X, Z)AX - g(AX, X)\phi AZ) \\ + (c/4)(g(\phi X, Z)AX + g(AX, Z)\phi X - 2\eta(Z)g(X, A\phi X)\xi).$$

Hence

$$(R(X, \phi X) \cdot A)Z = (c/4)(g(X, Z)\phi AX + g(X, \phi AZ)X - g(X, \phi Z)AX \\ + g(X, AZ)\phi X).$$

Now let $\{e_i\}$ be an orthonormal basis of $\xi^\perp$. Then we have

$$\sum (R(e_i, \phi e_i) \cdot A)Z = (c/4)(\phi AZ + \phi AZ - A\phi Z + \phi AZ) \\ = c\phi AZ. \tag{2.28}$$

We consider the following for any (1, 1) tensor T

$$(TX \wedge T\phi X)AZ - A(TX \wedge T\phi X)Z,$$

where $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$. In general, we find

$$(TX \wedge T\phi X)AZ - A(TX \wedge T\phi X)Z \\ = g(T\phi X, AZ)TX - g(TX, AZ)T\phi X - g(T\phi X, Z)ATX \\ + g(TX, Z)AT\phi X. \tag{2.29}$$

Summing (2.29) over $X = e_i$, we can see that the right hand side becomes

$$-T(\phi T^* AZ) - T\phi T^* AZ + AT\phi T^* Z + AT\phi T^* Z = -2T\phi T^* AZ + 2AT\phi T^* Z, \tag{2.30}$$

where T^* is the transpose of T. In the case of $T = I$, (2.30) becomes

$$2(A\phi - \phi A)Z = -4\phi AZ.$$

When $T = A$, (2.30) is

$$-2A\phi A^2 Z + 2A^2\phi AZ = -2(A\phi A)AZ + 2A(A\phi A)Z = -c\phi AZ.$$

Here we have used $A\phi A = (c/4)\phi$. It follows from (2.5) that

$$\begin{aligned} R(e_i, \phi e_i) &= Ae_i \wedge A\phi e_i + (c/4)(e_i \wedge \phi e_i + \phi e_i \wedge \phi^2 e_i + 2g(e_i, \phi^2 e_i)\phi) \\ &= Ae_i \wedge A\phi e_i + (c/2)(e_i \wedge \phi e_i) - (c/2)\phi. \end{aligned}$$

Since $(R(e_i, \phi e_i) \cdot A)Z = R(e_i, \phi e_i)(AZ) - AR(e_i, \phi e_i)Z$, by the summation of the last term in $(R(e_i, \phi e_i) \cdot A)Z$ gives $-c(2n - 2)\phi AZ$. Using this and (2.30) with $T = I$ and $T = A$, we see that

$$\sum (R(e_i, \phi e_i) \cdot A)Z = -c(2n + 1)\phi AZ. \tag{2.31}$$

For all tangent vectors Z, from (2.28) and (2.31) we find that

$$2c(n + 1)\phi AZ = 0,$$

so that $\phi A = 0$. This implies that $AX = \eta(AX)\xi$ for all $X \in TM$. Hence, from (2.4) we know that

$$(\nabla_X A)Y = \nabla_X (AY) - A\nabla_X Y \in \operatorname{span}\{\xi\},$$

which, together with (2.6), yields

$$(c/4)(\eta(X)\phi Y - \eta(Y)\phi X) \in \operatorname{span}\{\xi\}.$$

Putting $Y = \xi$ in this equation, we can see that $(-c/4)\phi X \in \operatorname{span}\{\xi\}$ for all $X \in TM$. Thus we obtain a contradiction. Therefore, in the case of $c < 0$ we conclude that δ is constant locally on M. □

The discussion in the proof of Lemma 1 gives the following:

Lemma 2. *There exist no real hypersurfaces with $\phi A + A\phi = 0$ in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$.*

Proof. Suppose that there exists a real hypersurface M with $\phi A + A\phi = 0$ in $\widetilde{M}_n(c)$. Then $\phi A\xi = -A\phi\xi = 0$, which implies that our real hypersurface M is a Hopf hypersurface. However, the discussion in the proof of Lemma 1 shows that there does not exist a Hopf hypersurface with $\phi A + A\phi = 0$ in a nonflat complex space form. □

By virtue of Lemma 2 we obtain the following fundamental property of all Hopf hypersurfaces in a nonflat complex space form.

Proposition 1. *For every Hopf hypersurface M in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$, the holomorphic distribution $T^0M = \{X \in TM | \eta(X) = 0\}$ on M is not integrable.*

Proof. Suppose that there exist a Hopf hypersurface M with the integrable holomorphic distribution T^0M in $\widetilde{M}_n(c)$. Note that T^0M is integrable if and only if

$$[X,Y] = \nabla_X Y - \nabla_Y X \in T^0M \quad \text{for all } X, Y \in T^0M.$$

Hence, for all $X, Y \in T^0M$ from (2.4) we have

$$\begin{aligned} 0 &= g(\nabla_X Y - \nabla_Y X, \xi) = -g(Y, \nabla_X \xi) + g(X, \nabla_Y \xi) \\ &= -g(Y, \phi AX) + g(X, \phi AY) = g(X, (\phi A + A\phi)Y), \end{aligned}$$

which implies that T^0M is integrable if and only if

$$g((\phi A + A\phi)X, Y) = 0 \quad \text{for all } X, Y \in T^0M. \tag{2.32}$$

This, combined with the assumption that ξ is principal, shows that $\phi A + A\phi = 0$ holds on our real hypersurface M. This is a contradiction. $\square$

We see easily that a real hypersurface M is a Hopf hypersurface if and only if every integral curve of ξ is a geodesic on M (see (2.4)). The following gives another characterization of all Hopf hypersurfaces in a nonflat complex space form.

Proposition 2. *For a real hypersurface M in a nonflat complex space form $\widetilde{M}_n(c)$, $n \geqq 2$ the following two conditions are mutually equivalent.*

1. *M is a Hopf hypersurface in $\widetilde{M}_n(c)$.*
2. *At each point $x \in M$, if we take a totally geodesic complex curve $\widetilde{M}_1(c)$ in $\widetilde{M}_n(c)$ through x whose tangent space $T_x\widetilde{M}_1(c)$ is the complex one dimensional linear subspace of $T_x\widetilde{M}_n(c)$ spanned by ξ_x, then the normal section $N_x = M \cap \widetilde{M}_1(c)$ given by $\widetilde{M}_1(c)$ is the integral curve through the point x of the characteristic vector field ξ of M.*

Proof. It follows from (2.1) and (2.4) that $\widetilde{\nabla}_\xi \xi = \phi A\xi + g(A\xi, \xi)\mathcal{N}$. This equation shows that Condition(1) in our Proposition is equivalen to

$$\widetilde{\nabla}_\xi \xi = g(A\xi, \xi)\mathcal{N} = g(A\xi, \xi)J\xi,$$

which is nothing but Condition (2). $\square$

3. Homogenous real hypersurfaces in $\mathbb{C}P^n(c)$

Takagi ([14]) classified all homogeneous real hypersurfaces in $\mathbb{C}P^n(c)$, namely they are orbits of some subgroups of the full asometry group $\mathrm{I}(\mathbb{C}P^n(c))$.

The following list is the so-called *Takagi's list.*

Theorem A ([14, 8])**.** *In $\mathbb{C}P^n(c)$ $(n \geqq 2)$, a homogeneous real hypersurface is locally congruent to one of the following Hopf hypersurfaces all of whose principal curvatures are constant*:

- (A_1) *A geodesic sphere of radius r, where $0 < r < \pi/\sqrt{c}$;*
- (A_2) *A tube of radius r around a totally geodesic $\mathbb{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \pi/\sqrt{c}$;*
- (B) *A tube of radius r around a complex hyperquadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c}\,)$;*
- (C) *A tube of radius r around the Segre embedding of $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and n $(\geqq 5)$ is odd*;

(D) *A tube of radius r around the Plüker embedding of a complex Grassmannian $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 9$;*
(E) *A tube of radius r around a Hermitian symmetric space $\mathrm{SO}(10)/\mathrm{U}(5)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 15$.*

These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). Unifying real hypersurfaces of types (A_1) and (A_2), we call them hypersurfaces of type (A). The numbers of distinct principal curvatures of these real hypersurfaces are $2, 3, 3, 5, 5, 5$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}P^n(c)$ are given as follows:

	(A_1)	(A_2)	(B)	(C, D, E)
λ_1	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r\big)$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r\big)$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\big)$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\big)$
λ_2	—	$-\frac{\sqrt{c}}{2}\tan\big(\frac{\sqrt{c}}{2}r\big)$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\big)$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\big)$
λ_3	—	—	—	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r\big)$
λ_4	—	—	—	$-\frac{\sqrt{c}}{2}\tan\big(\frac{\sqrt{c}}{2}r\big)$
δ	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$

One should notice that in $\mathbb{C}P^n(c)$ a tube of radius r $(0 < r < \pi/\sqrt{c}\,)$ around a totally geodesic $\mathbb{C}P^\ell(c)$ $(0 \leqq \ell \leqq n-1)$ is congruent to a tube of radius $\big((\pi/\sqrt{c}\,) - r\big)$ around a totally geodesic $\mathbb{C}P^{n-\ell-1}(c)$. So, in particular by setting $\ell = 0$ we know that a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ is congruent to a tube of radius $\big((\pi/\sqrt{c}\,) - r\big)$ around a totally geodesic hypersurface $\mathbb{C}P^{n-1}(c)$. Then we can see that all homogeneous real hypersurfaces of $\mathbb{C}P^n(c)$ are realized as tubes of constant radius around compact Hermitian symmetric spaces of rank 1 or 2. The multiplicities of these principal curvatures are given as follows (cf. [15]):

	(A_1)	(A_2)	(B)	(C)	(D)	(E)
$m(\lambda_1)$	$2n-2$	$2n-2\ell-2$	$n-1$	2	4	6
$m(\lambda_2)$	—	2ℓ	$n-1$	2	4	6
$m(\lambda_3)$	—	—	—	$n-3$	4	8
$m(\lambda_4)$	—	—	—	$n-3$	4	8
$m(\delta)$	1	1	1	1	1	1

By the above tables of the principal curvatures we find easily a minimal homogeneous real hypersurface in $\mathbb{C}P^n(c)$.

Lemma 3. *A homogeneous real hypersurface of type either* $(\mathrm{A}_1), (\mathrm{A}_2)$, $(\mathrm{B}), (\mathrm{C}), (\mathrm{D})$ *or* (E)*, which is a tube of radius r, is minimal in the following cases*:
(A_1) $\cot(\sqrt{c}\,r/2) = 1/\sqrt{2n-1}\,$;
(A_2) $\cot(\sqrt{c}\,r/2) = \sqrt{(2\ell+1)/(2n-2\ell-1)}\,$;

(B) $\cot(\sqrt{c}\, r/2) = \sqrt{n} + \sqrt{n-1}$;
(C) $\cot(\sqrt{c}\, r/2) = (\sqrt{n} + \sqrt{2}\,)/\sqrt{n-2}$;
(D) $\cot(\sqrt{c}\, r/2) = \sqrt{5}$;
(E) $\cot(\sqrt{c}\, r/2) = (\sqrt{15} + \sqrt{6}\,)/3$.

Proof. Let M be of type (A_1). Setting Trace $A = 0$, we have

$$(2n-2)\frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}r\Big) + \sqrt{c}\,\cot(\sqrt{c}\, r) = 0.$$

Here we note that $2\cot 2\theta = \cot\theta - \tan\theta$. Then, putting $x = \cot(\sqrt{c}\, r/2)$, we get

$$(2n-2)x + x - \frac{1}{x} = 0,$$

so that $x^2 = 1/(2n-1)$. Since $x > 0$, we see

$$\cot\Big(\frac{\sqrt{c}}{2}r\Big) = \frac{1}{\sqrt{2n-1}}.$$

Let M be of type (A_2). It follows from Trace $A = 0$ that

$$(2n-2\ell-2)\frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}r\Big) - 2\ell\frac{\sqrt{c}}{2}\tan\Big(\frac{\sqrt{c}}{2}r\Big) + \sqrt{c}\cot(\sqrt{c}r) = 0.$$

Hence we obtain

$$(2n-2\ell-2) - \frac{2\ell}{x} + x + \frac{1}{x} = 0,$$

so that

$$x^2 = \frac{2\ell+1}{2n-2\ell-1}.$$

Thus we find

$$\cot\Big(\frac{\sqrt{c}}{2}r\Big) = \sqrt{\frac{2\ell+1}{2n-2\ell-1}}.$$

Let M be of type (B). It follows from Trace $A = 0$ that

$$(n-1)\frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\Big) + (n-1)\frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\Big) + \sqrt{c}\cot(\sqrt{c}r) = 0.$$

So we get

$$(n-1)\frac{x+1}{1-x} + (n-1)\frac{x-1}{1+x} + x - \frac{1}{x} = 0,$$

which implies

$$x^4 - 2(2n-1)x + 1 = 0.$$
$$x^2 = 2n-1 \pm \sqrt{n(n-1)}.$$

Since $x > 1$, we see

$$\cot\Big(\frac{\sqrt{c}}{2}r\Big) = \sqrt{n} + \sqrt{n-1}.$$

Let M be of type (C). Setting Trace $A = 0$, we have

$$2\cdot\frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\Big) + 2\cdot\frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\Big) + (n-3)\frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}r\Big)$$
$$- (n-3)\frac{\sqrt{c}}{2}\tan\Big(\frac{\sqrt{c}}{2}r\Big) + \sqrt{c}\cot(\sqrt{c}\, r) = 0.$$

So we get

$$2 \cdot \frac{x+1}{1-x} + 2 \cdot \frac{x-1}{1+x} + (n-3)x - (n-3)\frac{1}{x} + x + \frac{1}{x} = 0,$$

which shows

$$\begin{aligned} 2x(1+x)^2 - &2x(1-x)^2 + (n-3)x^2(1-x^2) - (n-3)(1-x^2) \\ &+ x^2(1-x^2) - (1-x^2) = 0. \end{aligned}$$

Hence we find

$$2x \cdot 4x - (n-3)(1-x^2)^2 - (1-x^2)^2 = 0,$$

so that

$$(n-2)x^4 - 2(n+2)x^2 + (n-2) = 0.$$

Thus we have

$$x^2 = \frac{n+2 \pm 2\sqrt{2n}}{n-2} = \frac{(\sqrt{n} \pm \sqrt{2}\,)^2}{n-2}.$$

This, together with $x > 1$, yields

$$\cot\Big(\frac{\sqrt{c}}{2}r\Big) = \frac{\sqrt{n}\, + \sqrt{2}}{\sqrt{n-2}}.$$

Let M be of type (D). Setting Trace $A = 0$, we have

$$\begin{aligned} 4 \cdot \frac{\sqrt{c}}{2} \cot\Big(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\Big) + 4 \cdot \frac{\sqrt{c}}{2} \cot\Big(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\Big) + 4\frac{\sqrt{c}}{2} \cot\Big(\frac{\sqrt{c}}{2}r\Big) \\ - 4\frac{\sqrt{c}}{2} \tan\Big(\frac{\sqrt{c}}{2}r\Big) + \sqrt{c}\cot(\sqrt{c}\, r) = 0. \end{aligned}$$

So we get

$$4 \cdot \frac{x+1}{1-x} + 4 \cdot \frac{x-1}{1+x} + 4x - 4\frac{1}{x} + x + \frac{1}{x} = 0,$$

which shows

$$\begin{aligned} 4x(1+x)^2 - &4x(1-x)^2 + 4x^2(1-x^2) - 4(1-x^2) \\ &+ x^2(1-x^2) - (1-x^2) = 0. \end{aligned}$$

Hence we find

$$4x \cdot 4x - 4(1-x^2)^2 - (1-x^2)^2 = 0,$$

so that

$$5x^4 - 26x^2 + 5 = 0.$$

Thus we have

$$x^2 = \frac{1}{5}, 5.$$

This, together with $x > 1$, yields

$$\cot\Big(\frac{\sqrt{c}}{2}r\Big) = \sqrt{5}\,.$$

Let M be of type (E). Setting Trace $A=0$, we have

$$6\cdot\frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}r-\frac{\pi}{4}\Big)+6\cdot\frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}r+\frac{\pi}{4}\Big)+8\frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}r\Big)$$
$$-8\frac{\sqrt{c}}{2}\tan\Big(\frac{\sqrt{c}}{2}r\Big)+\sqrt{c}\cot(\sqrt{c}\,r)=0.$$

So we get

$$6\cdot\frac{x+1}{1-x}+6\cdot\frac{x-1}{1+x}+8x-8\frac{1}{x}+x+\frac{1}{x}=0,$$

which shows

$$6x(1+x)^2-6x(1-x)^2+8x^2(1-x^2)-8(1-x^2)$$
$$+x^2(1-x^2)-(1-x^2)=0.$$

Hence we find

$$6x\cdot 4x-8(1-x^2)^2-(1-x^2)^2=0,$$

so that

$$3x^4-14x^2+3=0.$$

Thus we have

$$x^2=\frac{7\pm2\sqrt{10}}{3}=\frac{(\sqrt{5}\pm\sqrt{2}\,)^2}{3}.$$

This, together with $x>1$, yields

$$\cot\Big(\frac{\sqrt{c}}{2}r\Big)=\frac{\sqrt{15}+\sqrt{6}}{3}.$$

$\square$

4. Classification of homogeneous Hopf hypersurfaces in $\mathbb{C}H^n(c)$

It is known that all homoogeneous real hypersurfaces M in $\mathbb{C}P^n(c)$ are Hopf hypersurfaces all of whose principal curvatures are constant on M in the ambient space $\mathbb{C}P^n(c)$. Inspired by this fact, Berndt ([4]) classified all Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n(c)$.

Theorem B ([4])**.** *Let M be a connected Hopf hypersurface all of whose principal curvatures are constant in $\mathbb{C}H^n(c)$ $(n\geqq 2)$. Then M is locally congruent to one of the following*:

- (A_0) *A horosphere in $\mathbb{C}H^n(c)$;*
- ($A_{1,0}$) *A geodesic sphere of radius r $(0<r<\infty)$;*
- ($A_{1,1}$) *A tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0<r<\infty$;*
- (A_2) *A tube of radius r around a totally geodesic $\mathbb{C}H^{\ell}(c)$ $(1\leqq\ell\leqq n-2)$, where $0<r<\infty$;*
- (B) *A tube of radius r around a totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0<r<\infty$.*

The real hypersurfaces in Theorems 4 are said to be of types (A_0), (A_1), (A_1), (A_2) and (B). Here, type (A_1) means either type ($A_{1,0}$) or type ($A_{1,1}$). Unifying real hypersurfaces of types (A_0), (A_1) and (A_2), we call them hypersurfaces of type (A). A real hypersurface of type (B) with radius $r=(1/\sqrt{|c|}\,)\log_e(2+\sqrt{3}\,)$ has two distinct constant principal curvatures $\lambda_1=\delta=\sqrt{3|c|}/2$ and $\lambda_2=\sqrt{|c|}/(2\sqrt{3}\,)$.

Except for this real hypersurface, the numbers of distinct principal curvatures of Hopf hypersurfaces with constant principal curvatures are $2, 2, 2, 3, 3$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}H^n(c)$ are given as follows (see [4]):

	(A_0)	($A_{1,0}$)	($A_{1,1}$)	(A_2)	(B)
λ_1	$\frac{\sqrt{\lvert c\rvert}}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$
λ_2	—	—	—	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$
δ	$\sqrt{\lvert c\rvert}$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\tanh(\sqrt{\lvert c\rvert}r)$

The multiplicities of these principal curvatures are given as follows (see [4]):

	(A_0)	($A_{1,0}$)	($A_{1,1}$)	(A_2)	(B)
$m(\lambda_1)$	$2n-2$	$2n-2$	$2n-2$	$2n-2\ell-2$	$n-1$
$m(\lambda_2)$	—	—	—	2ℓ	$n-1$
$m(\delta)$	1	1	1	1	1

Berndt and Tamaru ([5]) classified all homogeneous real hypersurfaces in $\mathbb{C}H^n(c)$. Due to their theorem we can see that there exist many homogeneous real hypersurfaces which are *not* Hopf hypersurfaces in $\mathbb{C}H^n(c)$.

5. Hypersurfaces of type (A) in $\widetilde{M}_n(c)$

We first recall the differential equation of the shape operator A of a hypersurface of type (A) in $\widetilde{M}_n(c)$.

Lemma 4 ([12, 13]). *Let M^{2n-1} be a connected real hypersurface in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$. Then M is locally congruent to a hypersurface of type* (A) *if and only if the shape operator A of M satisfies*

$$(\nabla_X A)Y = -(c/4)(g(\phi X, Y)\xi + \eta(Y)\phi X) \quad \textit{for } X, Y \in TM. \tag{5.1}$$

By virtue of Lemma 4 we get the following fundamental result.

Theorem C ([12, 13]). *Let M^{2n-1} be a real hypersurface in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$. Then the length of the derivative of the shape operator A of M satisfies $\|\nabla A\|^2 \geqq (c^2/4)(n-1)$. In particular, $\|\nabla A\|^2 = (c^2/4)(n-1)$ holds on M if and only if M is locally congruent to a hypersurface of type* (A).

Proof. We set the tensor field of type $(1, 2)$ on M given by

$$T(X, Y) = (\nabla_X A)Y + (c/4)(g(\phi X, Y)\xi + \eta(Y)\phi X).$$

We take a local field of orthonormal frames $\{e_1, e_2, \ldots, e_{2n-1}\}$ on M. In order to compute the norm $\|T\|$ of T, we have the following equalities. Using $\phi^2 = -I + \eta\otimes\xi$, $\eta(\xi) = 1$ and $g(\xi, \xi) = 1$, we have

$$\frac{c^2}{16}\sum_{i,j}\big(g(\phi e_i, e_j)\big)^2 = \frac{c^2}{16}\sum_{i,j} g(\eta(e_j)\phi e_i, \eta(e_j)\phi e_i) = \frac{c^2}{8}(n-1). \tag{5.2}$$

Next, using the symmetry of A, skew-symmetry of ϕ and Codazzi equation (2.6), we find

$$\begin{aligned}(5.3)\qquad \frac{c}{2}\sum_{i,j} g(\phi e_i, e_j)g((\nabla_{e_i}A)e_j, \xi) &= \frac{c}{2}\sum_i g(\phi e_i, (\nabla_{e_i}A)\xi)\\ &= \frac{c}{2}\sum_i g(\phi e_i, (\nabla_\xi A)e_i - (c/4)\phi e_i)\\ &= \frac{c}{2}\mathrm{tr}((\nabla_\xi A)\phi) - \frac{c^2}{8}\sum_i g(\phi e_i, \phi e_i) = -\frac{c^2}{4}(n-1).\end{aligned}$$

Similarly, we get

$$(5.4)\qquad \frac{c}{2}\sum_{i,j} \eta(e_j)g((\nabla_{e_i}A)e_j, \phi e_i) = -\frac{c^2}{4}(n-1).$$

Thus, from (5.2), (5.3) and (5.4) we obtain the desired inequality. □

Theorem C implies the importance of hypersurfaces of type (A) in a nonflat complex space form.

At the end of this section we recall the following:

Remark 1. For a real hypersurface m^{2n-1} in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$, M is of type (A) if and only if $\phi A = A\phi$ holds on M, where ϕ and A are the structure tensor and A the shpape operator of M, respectively (for example, see [13]).

6. Main Result

First of all we recall the definition of circles in Riemannian geometry. A smooth real curve $\gamma = \gamma(s)$ $(s \in I)$ in a Riemannian manifold M with Riemmanian metric g is called a *circle* of curvature k if the ordinary differential equations $\nabla_{\dot\gamma}\dot\gamma = kY_s$ and $\nabla_{\dot\gamma}Y_s = -k\dot\gamma$ hold for each $s \in I$, where $\nabla_{\dot\gamma}$ is the covariant differentiation along γ with respect to the Riemannian connection ∇ of M and k is a nonnegative constant. A circle of null curvature is nothing but a geodesic. The definition of circles can be rewritten as follows: A smooth real curve $\gamma = \gamma(s)$ $(s \in I)$ in a Riemannian manifold M is called a circle if it satisfies the ordinary differential equation

$$(6.1)\qquad \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma)\dot\gamma = 0.$$

The following is fundamental.

Proposition 3. *For a connected hypersurface M^n isometrically immersed into a Riemannian manifold $\widetilde{M}^{n+1}$, the following three conditions are mutually equivalent*:

1. *Every geodesic γ on M^n is mapped to a circle in $\widetilde{M}^{n+1}$;*
2. *Every geodesic γ on M^n is mapped to a circle of the same curvature which is independent of the choice of γ in $\widetilde{M}^{n+1}$;*
3. *M^n is totally umbilic in $\widetilde{M}^{n+1}$ and M^n has constant mean curvature, namely* Trace *A is constant on M^n, where A is the shape operator of M^n in $\widetilde{M}^{n+1}$.*

Proof. We suppose Condition (1). Then, from (6.1) every geodesic γ of M^n, considered as a curve in the ambient space $\widetilde{M}^{n+1}$, satisfies the following ordinary differential equation:

$$\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma + g(\widetilde{\nabla}_{\dot\gamma}\dot\gamma, \widetilde{\nabla}_{\dot\gamma}\dot\gamma)\dot\gamma = 0. \tag{6.2}$$

On the other hand, in consideration of Gauss formula: $\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}$ and Weingarten formula: $\widetilde{\nabla}_X\mathcal{N} = -AX$ for the hypersurface M^n in $\widetilde{M}^{n+1}$, we can rewrite (6.2) as follows:

$$-g(A\dot\gamma, \dot\gamma)A\dot\gamma + g(A\dot\gamma, \dot\gamma)^2\dot\gamma + g((\nabla_{\dot\gamma}A)\dot\gamma, \dot\gamma)\mathcal{N} = 0. \tag{6.3}$$

Hence, taking the tangential component and the normal component of (6.3) for the hypersurace M^n in $\widetilde{M}^{n+1}$, we obtain

$$g(A\dot\gamma, \dot\gamma)A\dot\gamma = g(A\dot\gamma, \dot\gamma)^2\dot\gamma \quad \text{and} \quad g((\nabla_{\dot\gamma}A)\dot\gamma, \dot\gamma) = 0 \tag{6.4}$$

for each geodesic γ on M^n. Equation (6.4) means that

$$g(AX, X)AX = g(AX, X)^2X \quad \text{and} \quad g((\nabla_X A)X, X) = 0 \tag{6.5}$$

for all $X \in TM$ with $\|X\| = 1$. Note that the former equation in (6.5) means

$$g(AX, X)g(AX, Y) = 0 \tag{6.6}$$

for each pair of orthonormal vectors X and Y on M, which is equivalent to saying that

$$g(A_pX, X)^2 \text{ is constant at each point } p \in M \tag{6.7}$$

for every unit vector $X \in T_pM$.

Indeed, let $f : S^{n-1}(1)(\subset \mathbb{R}^n) \to \mathbb{R}$ be the differentiable function on a subset $S^{n-1}(1) \cong \{u \in T_pM \mid \|u\| = 1\}$ defined by $f(u) = g(A_pu, u)^2$, where A_p is the shape operator of M in $\widetilde{M}^{n+1}$ at the point $p \in M$. If v is a vector tangent to $S^{n-1}(1)$ at u (hence $u \perp v$), we find $v(f) = 4g(A_pu, u)g(A_pu, v) = 0$ by (6.6). Thus f is a constant function on $S^{n-1}(1)$.

Then we can set $\lambda^2(p) = g(AX, X)^2$ for each unit vector $X \in T_pM$ with $\lambda(p) \geqq 0$ at every point $p \in M$. When M^n is not totally geodesic in $\widetilde{M}^{n+1}$, there exists a point $x \in M$ with $\lambda(x) > 0$. Then the continuity of the function λ shows that there exists some open neighborhood U_x of the point x such that $\lambda > 0$ on U_x. We here choose a local field of orthonormal frames $e_1, \ldots, e_n$ on U_x in such a way that $Ae_i = \lambda_i e_i$ $(1 \leqq i \leqq n)$. Hence, from (6.7) we see that $\lambda_1^2 = \ldots = \lambda_n^2 = \lambda^2$. In this case, we suppose that there exist an orthonormal pair of vectors e_i and e_j such that $Ae_i = \lambda e_i$ and $Ae_j = -\lambda e_j$. Then we find that

$$g(A(e_i + e_j)/\sqrt{2}, (e_i + e_j)/\sqrt{2}\,) = 0,$$

which is a contradiction. So, we know that $Ae_i = \lambda e_i$ $(1 \leqq i \leqq n)$, which shows that every point $y \in U_x$ is an umbilic point. Thus we can see that M^n is totally umbilic in $\widetilde{M}^{n+1}$. Furthermore, the latter equation in (6.5) yields that the function λ is constant on M. Therefore we get Conditions (2) and (3) in our Proposition.

By virtue of the above argument in the proof of our Proposition we can see that each of Conditions (2) and (3) implies Condition (1). $\square$

As an immediate consequence of Proposition 3 we get

Lemma 5. *Let M^n be a hypersurface isometrically immersed into a Riemannian manifold $\widetilde{M}^{n+1}$. If a geodesic $\gamma = \gamma(s)$ $(s \in I)$ on M is mapped to a circle of positive curvature k, then the shape operator A of M^n in $\widetilde{M}^{n+1}$ satisfies either $A\dot{\gamma}(s) = k\dot{\gamma}(s)$ for all $s \in I$ or $A\dot{\gamma}(s) = -k\dot{\gamma}(s)$ for all $s \in I$.*

On the other hand, we recall the following:

Proposition 4 ([16]). *There exist no totally umbilic real hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$.*

Proof. Suppose that $AX = \lambda X$ for all vectors X on M. For any vectors X and Y orthogonal to ξ, from the Codazzi equation (2.6) we have $(X\lambda)Y - (Y\lambda)X = (c/2)g(X, \phi Y)\xi$, so that $g(X, \phi Y) = 0$ for all X, Y perpendicular to ξ. This is a contradiction. $\square$

Motivated by Propositions 3 and 4, for a real hypersurface M^{2n-1} in $\mathbb{C}P^n(c)$ we shall consider the condition that some of whose geodesics are mapped to circles of the same positive curvature in this ambient space.

We are now in a position to prove the following main theorem.

Theorem. *A connected minimal real hypersurface M^{2n-1} of $\mathbb{C}P^n(c), n \geqq 3$ is locally congruent to a tube of radius $r = (2/\sqrt{c}\,)\cot^{-1}\sqrt{(2\ell+1)/(2n-2\ell-1)}$ $(0 < r < \pi/\sqrt{c}\,)$ around a totally geodesic $\mathbb{C}P^\ell(c)$ with $1 \leqq \ell \leqq n-2$ if and only if there exist a function $d : M \to \mathbb{N}$ and orthonormal vectors $v_1, v_2, \ldots, v_{2n-2}$ perpendicular to the characteristic vector ξ_x at each point $x \in M$ satisfying the following two conditions*:

- *(i) All geodesics $\gamma_i = \gamma_i(s)$ on M^{2n-1} with $\gamma_i(0) = x$ and $\dot{\gamma}_i(0) = v_i$ $(1 \leqq i \leqq 2n-2)$ are mapped to circles of positive curvature in $\mathbb{C}P^n(c)$;*
- *(ii) All geodesics $\gamma_{ij} = \gamma_{ij}(s)$ on M^{2n-1} with $\gamma_{ij}(0) = x$ and $\dot{\gamma}_{ij}(0) = av_i + \sqrt{1-a^2}\,v_j$ $(1 \leqq i \leqq d_x < j \leqq 2n-2)$ are mapped to geodesics in $\mathbb{C}P^n(c)$, where $a = \sqrt{(2\ell+1)/(2n)}$.*

In this case, d is automatically expressed as $d = 2\ell$.

Proof. We first investigate the "only if" part of our Theorem. It is known that a real hypersurface M of type (A_2) with radius r $(0 < r < \pi/\sqrt{c}\,)$ has three distinct constant principal curvatures $\lambda_1 = (-\sqrt{c}/2)\tan(\sqrt{c}\,r/2), \lambda_2 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$ and $\delta = \sqrt{c}\cot(\sqrt{c}\,r) = \lambda_1 + \lambda_2$. As our real hypersurface M of type (A_2) is minimal, the principal curvatures λ_1 and λ_2 are expressed as follows (see Lemma 3):

$$\lambda_1 = -\frac{\sqrt{c}}{2}\sqrt{\frac{2n-2\ell-1}{2\ell+1}} \quad \text{and} \quad \lambda_2 = \frac{\sqrt{c}}{2}\sqrt{\frac{2\ell+1}{2n-2\ell-1}}. \tag{6.8}$$

Take orthonormal vectors $v_1, v_2, \ldots, v_{2n-2}$ orthogonal to ξ at an arbitrary point x of M in such a way that $v_1, v_2, \ldots, v_{2\ell}$ and $v_{2\ell+1}, \ldots, v_{2n-2}$ are principal curvature vectors with principal curvatures λ_1 and λ_2, respectively. Then by virtue of Lemma in [10] we find that these vectors satisfy Condition (i). That is, we have the following:

- (i) All geodesics $\gamma_i = \gamma_i(s)$ on M with $\gamma_i(0) = x$ and $\dot{\gamma}_i(0) = v_i$ $(1 \leqq i \leqq 2\ell)$ are circles of positive curvature $|\lambda_1|$ in $\mathbb{C}P^n(c)$;
- (ii) All geodesics $\gamma_i = \gamma_i(s)$ on M with $\gamma_i(0) = x$ and $\dot{\gamma}_i(0) = v_i$ $(2\ell + 1 \leqq i \leqq 2n-2)$ are circles of positive curvature λ_2 in $\mathbb{C}P^n(c)$.

We next take the geodesic $\gamma_{ij} = \gamma_{ij}(s)$ on M^{2n-1} with $\gamma_{ij}(0) = x$ and $\dot\gamma_{ij}(0) = av_i + \sqrt{1-a^2}\,v_j$ $(1 \leqq i \leqq d_x = 2\ell < j \leqq 2n-2)$, where $a = \sqrt{(2\ell+1)/(2n)}$. It is well-known that the shape operator A of our real hypersurface M satisfies (cf. [12]):

$$g((\nabla_X A)X, X) = 0 \quad \text{for each } X \in TM. \tag{6.9}$$

It follows from (2.1), (6.8) and (6.9) that

$$\begin{aligned} g(\widetilde\nabla_{\dot\gamma_{ij}}\dot\gamma_{ij}, \mathcal{N}) &= g(A\dot\gamma_{ij}(s), \dot\gamma_{ij}(s)) = g(A\dot\gamma_{ij}(0), \dot\gamma_{ij}(0)) \\ &= a^2\lambda_1 + (1-a^2)\lambda_2 = 0, \end{aligned}$$

which yields Condition (ii).

We shall investigate the "if" part of our Theorem. We consider a connected real hypersurface M^{2n-1} satisfying Conditions (i) and (ii). We explain the discussion in [1] in detail. We first concentrate our attention on Condition (i). We study on an open dense subset

$$\mathcal{U} = \left\{ x \in M^{2n-1} \,\middle|\, \begin{array}{l} \text{the multiplicity of each principal curvature of } M^{2n-1} \text{ in} \\ \mathbb{C}P^n(c) \text{ is constant on some neighborhood } \mathcal{V}_x(\subset \mathcal{U}) \text{ of } x \end{array} \right\}$$

of M^{2n-1}. We take the geodesic $\gamma_i = \gamma_i(s)$ $(1 \leqq i \leqq 2n-2)$ on $\mathcal{U}$ with initial vector v_i given by Condition (i). Since the curve γ_i, considered as a curve in $\mathbb{C}P^n(c)$, is a circle of positive curvature (, say) k_i, Equation (6.1) shows

$$\widetilde\nabla_{\dot\gamma_i}\widetilde\nabla_{\dot\gamma_i}\dot\gamma_i = -k_i^2\dot\gamma_i. \tag{6.10}$$

On the other hand, using (2.1) and (2.2), we see that

$$\widetilde\nabla_{\dot\gamma_i}\widetilde\nabla_{\dot\gamma_i}\dot\gamma_i = -g(A\dot\gamma_i, \dot\gamma_i)A\dot\gamma_i + g((\nabla_{\dot\gamma_i}A)\dot\gamma_i, \dot\gamma_i)\mathcal{N}. \tag{6.11}$$

Comparing the tangential components of Equations (6.10) and (6.11), we have

$$g(A\dot\gamma_i, \dot\gamma_i)A\dot\gamma_i = k_i^2\dot\gamma_i.$$

This, together with $k_i \neq 0$, shows that at $s = 0$ either $Av_i = k_iv_i$ or $Av_i = -k_iv_i$ holds for $i = 1, 2, \ldots, 2n-2$. This means that our real hypersurface M^{2n-1} is a Hopf hypersurface with $A\xi = \delta\xi$ and that the linear subspace $T_x^0M^{2n-1} = \{v \in T_xM^{2n-1} | v \perp \xi_x\}$ of T_xM^{2n-1} is decomposed as:

$$\begin{aligned} T_x^0M^{2n-1} &= \{v \in T_x^0M \mid Av = -k_{i_1}v\} \oplus \{v \in T_x^0M \mid Av = k_{i_1}v\} \\ &\quad \oplus \cdots \oplus \{v \in T_x^0M \mid Av = -k_{i_g}v\} \oplus \{v \in T_x^0M \mid Av = k_{i_g}v\}, \end{aligned}$$

where $0 < k_{i_1} < k_{i_2} < \ldots < k_{i_g}$ and g is the number of distinct positive k_i $(i = 1, \ldots, 2n-2)$. We decompose $T_x^0M^{2n-1}$ in such a way at each point $x \in \mathcal{U}$.

Note that each k_{i_j} is a smooth function on $\mathcal{V}_x$ for each $x \in \mathcal{U}$. We shall show the constancy of each k_{i_j}. It suffices to check the case of $Av_{i_j} = k_{i_j}v_{i_j}$. As k_{i_j} is a constant function along the curve γ_{i_j} in the ambient space $\mathbb{C}P^n(c)$, we have $v_{i_j}k_{i_j} = 0$.

In the following, we consider the case of $k_{i_j}(p) \neq \delta$. So we may suppose that $k_{i_j} \neq \delta$ at each point of a sufficiently small neighborhood of the point p. For any v_ℓ $(1 \leqq \ell \neq i_j \leqq 2n-2)$, since A is symmetric, we have

$$g((\nabla_{v_{i_j}}A)v_\ell, v_{i_j}) = g(v_\ell, (\nabla_{v_{i_j}}A)v_{i_j}). \tag{6.12}$$

In order to compute Equation (6.12) easily, we extend the vectors $v_\ell, v_{i_j} (\in T^0_x M)$ on some sufficiently small neighborhood $\mathcal{W}_x (\subset \mathcal{V}_x)$ in the following manner.

We define a smooth vector field V_ℓ on $\mathcal{W}_x$ satisfying that $(V_\ell)_x = v_\ell$ and V_ℓ is perpendicular to ξ. Next we shall define V_{i_j}. First we define a smooth unit vector field W_{i_j} on some "sufficiently small" neighborhood $\mathcal{W}_x (\subset \mathcal{V}_x)$ by using parallel displacement for the vector v_{i_j} along each geodesic with origin x. We note that in general W_{i_j} is not principal on $\mathcal{W}_x$, but $AW_{i_j} = k_{i_j} W_{i_j}$ on the geodesic $\gamma_{i_j} = \gamma_{i_j}(s)$ with $\gamma_{i_j}(0) = x$ and $\dot{\gamma}_{i_j}(0) = v_{i_j}$. We here define the vector field U_{i_j} on $\mathcal{W}_x$ as: $U_{i_j} = \Big(\prod_{\alpha \neq k_{i_j}} (A - \alpha I)\Big) W_{i_j}$, where α runs over the set of all distinct principal curvatures of M^{2n-1} except for the principal curvature k_{i_j}. We remark that $U_{i_j} \neq 0$ on the neighborhood $\mathcal{W}_x$, because $(U_{i_j})_x \neq 0$. Moreover, the vector field U_{i_j} satisfies $AU_{i_j} = k_{i_j} U_{i_j} (\perp \xi)$ on $\mathcal{W}_{i_j}$. We define V_{i_j} by normalizing U_{i_j} in some sense. That is, when $\prod_{\alpha \neq k_{i_j}} (k_{i_j} - \alpha)(x) > 0$ (resp. $\prod_{\alpha \neq k_{i_j}} (k_{i_j} - \alpha)(x) < 0$), we define $V_{i_j} = U_{i_j} / \|U_{i_j}\|$ (resp. $V_{i_j} = -U_{i_j} / \|U_{i_j}\|$). Then we know that $AV_{i_j} = k_{i_j} V_{i_j}$ on $\mathcal{W}_x$ and $(V_{i_j})_x = v_{i_j}$. Furthermore, our construction shows that the integral curve of V_{i_j} through the point x is a geodesic on M^n, so that in particular $\nabla_{V_{i_j}} V_{i_j} = 0$ at the point x.

Since the Codazzi equation (2.6) yields that $g((\nabla_X A)Y, Z) = g((\nabla_Y A)X, Z)$ for any $X, Y, Z (\perp \xi)$, at the point x we have

$$\begin{aligned}
(\text{the left-hand side of (6.12)}) &= g((\nabla_{v_\ell} A) v_{i_j}, v_{i_j}) \\
&= g((\nabla_{V_\ell} A) V_{i_j}, V_{i_j}) \\
&= g(\nabla_{V_\ell}(k_{i_j} V_{i_j}) - A \nabla_{V_\ell} V_{i_j}, V_{i_j}) \\
&= g((V_\ell k_{i_j}) V_{i_j} + (k_{i_j} I - A) \nabla_{V_\ell} V_{i_j}, V_{i_j}) \\
&= v_\ell k_{i_j}
\end{aligned}$$

and

$$\begin{aligned}
(\text{the right-hand side of (6.12)}) &= g(V_\ell, (\nabla_{V_{i_j}} A) V_{i_j}) \\
&= g(V_\ell, \nabla_{V_{i_j}}(k_{i_j} V_{i_j}) - A \nabla_{V_{i_j}} V_{i_j}) \\
&= g(v_\ell, (v_{i_j} k_{i_j}) v_{i_j}) = 0.
\end{aligned}$$

Thus we can see that $X k_{i_j} = 0$ for any $X (\perp \xi) \in T_x M$. Next, we shall show that $\xi k_{i_j} = 0$. It follows from (2.4) and Proposition A that

$$\begin{aligned}
(\nabla_\xi A) V_{i_j} - (\nabla_{V_{i_j}} A)\xi &= \nabla_\xi (A V_{i_j}) - A \nabla_\xi V_{i_j} - \nabla_{V_{i_j}}(\delta \xi) + A \nabla_{V_{i_j}} \xi \\
&= \nabla_\xi (k_{i_j} V_{i_j}) - A \nabla_\xi V_{i_j} - \delta \phi A V_{i_j} + A \phi A V_{i_j} \\
&= (\xi k_{i_j}) V_{i_j} + (k_{i_j} I - A) \nabla_\xi V_{i_j} - k_{i_j} \Big(\delta - \frac{\delta k_{i_j} + (c/2)}{2k_{i_j} - \delta}\Big) \phi V_{i_j}.
\end{aligned}$$

On the other hand, the Codazzi equation (2.6) implies

$$g((\nabla_\xi A) V_{i_j} - (\nabla_{V_{i_j}} A)\xi, V_{i_j}) = 0.$$

Hence, $\xi k_{i_j} = 0$. Therefore we can see that the differential dk_{i_j} of k_{i_j} vanishes at the point x, which shows that every k_{i_j} (> 0) is constant on $\mathcal{W}_x$, since we can take the point x as an arbitrarily fixed point of $\mathcal{W}_x$. So the principal curvature function k_{i_j} is locally constant on the open dense subset $\mathcal{U}$ of M^{2n-1}.

We next suppose that $2k_{i_j} - \delta = 0$ at some point $p \in \mathcal{U}$. Then, by the continuity of the principal curvature k_{i_j} and the local constancy of k_{i_j} in the above case there exists some sufficiently small neighborhood $\mathcal{W}_p$ of the point p such that $2k_{i_j} - \delta \equiv 0$ on $\mathcal{W}_p$. Hence, $k_{i_j} = \delta/2$ is constant on $\mathcal{W}_p$ (see Lemma 1).

Therefore, by the continuity of k_{i_j} and connectivity of M^{2n-1} we can see that k_{i_j} is constant on the hypersurface M^{2n-1}. Hence all the principal curvatures of M^{2n-1} are constant if M^{2n-1} satisfies Condition (i).

Next, we consider Condition (ii). Since the above argument tells us that every v_i $(1 \leqq i \leqq 2n-2)$ is principal, we can set $Av_i = \mu_i v_i$. On the other hand, Condition (ii) shows that $g(A\dot{\gamma}_{ij}(0), \dot{\gamma}_{ij}(0)) = 0$, so that

$$a^2\mu_i + (1-a^2)\mu_j = 0 \quad \text{for } 1 \leqq \forall i \leqq d_x < \forall j \leqq 2n-2. \tag{6.13}$$

This, combined with $0 < a^2 = (2\ell+1)/(2n) < 1$, implies that M is a Hopf hypersurface with three distinct constant principal curvatures δ, μ_i and μ_j satisfying Equation (6.13). Hence M is of either type (A_2) or type (B). Needless to say, all minimal real hypersurfaces of type (A_2) satisfy Equation (6.13) (see the "only if" part of the proof of our Theorem).

Finally we shall check the case of type (B). We know that a real hypersurface M of type (B) with radius r $(0 < r < \pi/(2\sqrt{c}\,))$ has three distinct constant principal curvatures $\lambda_1 = (\sqrt{c}/2)\cot\big((\sqrt{c}\,r)/2 - \pi/4\big)$, $\lambda_2 = (\sqrt{c}/2)\cot\big((\sqrt{c}\,r)/2 + \pi/4\big)$ and $\delta = \sqrt{c}\cot(\sqrt{c}\,r)$. As our real hypersurface M of type (B) is minimal, the principal curvatures λ_1 and λ_2 are expressed as (see Lemma 3):

$$\lambda_1 = -\frac{\sqrt{c}}{2}\frac{1+\sqrt{n}}{\sqrt{n-1}} \quad \text{and} \quad \lambda_2 = \frac{\sqrt{c}}{2}\frac{\sqrt{n}-1}{\sqrt{n-1}}. \tag{6.14}$$

The rest of the proof is to show that the principal curvatures λ_1 and λ_2 in (6.14) satisfy neither $a^2\lambda_1 + (1-a^2)\lambda_2 = 0$ nor $a^2\lambda_2 + (1-a^2)\lambda_1 = 0$. Suppose that $a^2\lambda_1 + (1-a^2)\lambda_2 = 0$. Then we have $-(2\ell+1)(1+\sqrt{n}\,) + (2n-2\ell-1)(\sqrt{n}-1) = 0$, so that $\sqrt{n} = n - 2\ell - 1$. Hence we can set $\sqrt{n} = p$ for some $p \in \mathbb{N}$, which implies that $p = p^2 - 2\ell - 1$. Thus we obtain the equality $p(p-1) = 2\ell + 1$, which is a contradiction. We next suppose that $a^2\lambda_2 + (1-a^2)\lambda_1 = 0$. By easy computation we get $(2\ell+1)(\sqrt{n}-1) - (2n-2\ell-1)(1+\sqrt{n}\,) = 0$, so that $-\sqrt{n} = n - 2\ell - 1$. Then by the same discussion as in the case of $\sqrt{n} = n - 2\ell - 1$ we also obtain a contradiction in this case. □

At the end of this paper we pose the following open problem:

Problem. *Characterize all homogeneous real hypersurfaces of type* (A_2) *in* $\mathbb{C}P^n(c)$ *from the viewpoint of our Theorem.*

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